CE382A

TRANSPORTATION ENGINEERING

Prof. Venkatesan Kanagaraj

Aman

- Traffic Jam is a traffic wave.
- Traffic Jam moves backward (sometimes forward also).
- Traffic Wave travels b/w −15 to −20 Km/h.
- Mathematical Theory of Traffic Flow.

Traffic Flow Theory

1) macroscopic View ⟶ collection of vehicles (aggregate)
2) Microscopic View ⟶ individual vehicle element

Traffic Variables :–

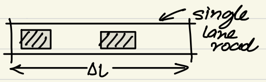
1) macroscopic variables → 1) Density (ρ), 2) Flow or Flux (Q), 3) Avg. vehicle speed (v)
2) microscopic variables

Macroscopic Variable

Density (ρ): no. of vehicles occupying given length of the road.

no. of vehicles ΔN

length of the road ΔL



$$\boxed{\rho = \frac{\Delta N}{\Delta L}}$$ (at a instant of time)

spatial quantity (not a temporal quantity)

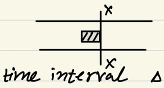
Unit: veh. /km / lane ⟶ For single lane, veh. /km

Flow (Q): No. of vehicles passing at particular point during specified time interval.

$$\boxed{Q = \frac{\Delta N}{\Delta t}}$$

→ Temporal Quantity

Unit: veh. /hour



time interval Δt

no. of vehicles passing x-x = ΔN

Avg. speed (v): $\boxed{v = \frac{1}{n}\sum v_i}$ ← individual vehicle speed.

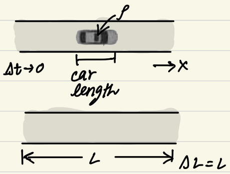
avg. veh. speed

Both Temporal and Spatial Quantity involved.

★ In macroscopic view, we assume that the traffic flow is similar to the fluid flow.

Continuum Approximation

Assume ρ, Q, v ⟶ smooth and continuous function each & every pt. in the road section



- $\rho = \lim_{\Delta L \to 0} \frac{\Delta N}{\Delta L}$ } ⟶ ΔL→0 is not possible

  Lots of fluctuations

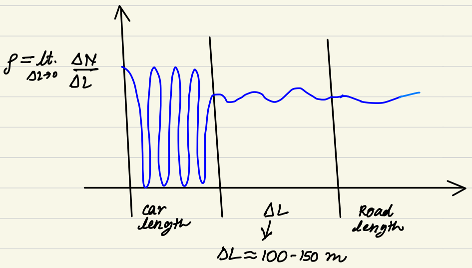
- ΔL ≃ car length, ρ=1, ρ=0 } ⟶ continuum hypothesis not valid ✗

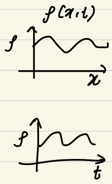
- ΔL ≃ road length } → continuum hypothesis not valid ✗

  ρ nearly constant

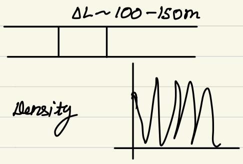
★ For $\boxed{\Delta L \simeq 100 - 150\, m}$ ⟶ Continuum Hypothesis holds !!

★ Vehicle Length ≪ ΔL ≪ Road Length ⟹ Then continuum hypothesis holds true !!





$\rho = \lim_{\Delta L \to 0} \frac{\Delta N}{\Delta L}$



ΔL = car length = 5

$\rho = \frac{1}{5}$ veh/m = 200 veh/Km ⟹ Jam Density

Empty Road ⟹ ρ = 0

ρ(x, t) → ordered traffic

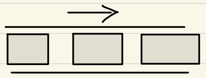
long. ↑ ↑ time

ρ(x, y, t) → Indian traffic → Disordered traffic (lane changing)

long. dirⁿ, lat. dirⁿ, time

Single lane Ordered traffic

Directional Traffic



one-direction

★ Extensive Quantity: They vary with no. of lanes

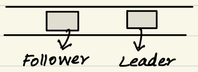
(ρ), (Q) → Extensive Quantity

★ Intensive Quantity: They do not vary with no. of lanes

Microscopic Variable

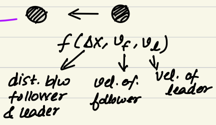
Newtonian Mechanics ⟶ We can't apply it here.

Because there is some delay in the force when break is applied by the leader.



Social Force

$$\frac{d^2x}{dt^2} = f(x, v_f, v_l)$$

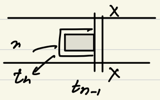


→ It refers to non physical factors that influence human behaviour and decision making.

① **Time Headaway** : Time difference between successive vehicles at a particular point

$$\boxed{h = t_n - t_{n-1}}$$

t_n ↓ Follower, t_{n-1} ↓ Leader



Total Observation Period $= \Delta T$

$h_{21} = t_2 - t_1$

$h_{23} = t_2 - t_3$

No. of vehicles crossing X X $= \Delta n$

Average time headaway, $\bar{h} = \frac{1}{\Delta n} \sum_{i=1}^{n} h_i$

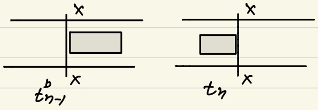
$\bar{h} = \frac{\Delta T}{\Delta n}$

$$\boxed{\bar{h} = \frac{1}{Q}} \rightarrow \left\{ Q = \frac{\Delta n}{\Delta T} \right\}$$

$\bar{h}$ → microscopic variable

Q → macroscopic variable

② **Time Gap** : Time Difference between back of front vehicle to front of following vehicle.



time gap $t_g = t_n - t_{n-1}$

$$\boxed{t_g = t_n - t_{n-1} - \frac{L_{n-1}}{v_{n-1}}}$$

$L_n \longrightarrow$ length of vehicle n

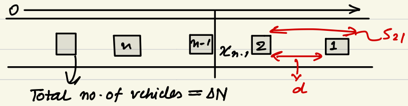
$v_n \rightarrow$ velocity of vehicle n

Eulerian & Lagrangian system
↓
Data collection

---

③ **Distance Headaway** : Distance between front bumber to front bumber of successive vehicle at a instant of time.

$$\boxed{S = x_{n-1} - x_n}$$



Total no. of vehicles $= \Delta N$
length of the road $= \Delta L$

Average Distance headaway $\bar{S} = \frac{1}{\Delta N} \Sigma s_i$

$\Sigma S_i = S_{21} + S_{32}$

$\Sigma s_i = \Delta L$

$\bar{S} = \frac{\Delta L}{\Delta N}$ $\qquad \rho = \frac{\Delta N}{\Delta L}$

$$\boxed{\bar{S} = \frac{1}{\rho}}$$

$\bar{S}$ → microscopic variable

ρ → macroscopic variable

④ **Distance Gap** : distance between back bumber of front vehicle to front bumber of following vehicle.

$$\boxed{d = x_{n-1} - x_n - L_{n-1}}$$

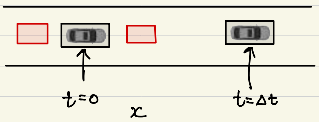
macro		micro	
Q		h	space fixed
ρ		t_g	
v	$\bar{h} = \frac{1}{Q}$	s	time fixed
		d	
	$\bar{S} = \frac{1}{\rho}$		

Traffic Data

① Lagrangian sensing Traffic Data. → observer inside the vehicle

② Eulerian sensing Traffic Data.

Lagrangian Description
or
Material Description



$v = \frac{dx}{dt}$; $a = \frac{d^2x}{dt^2}$

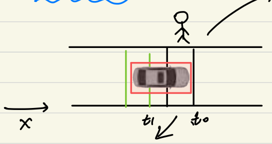
Two kinds of data

① Trajectory Data (All the vehicles) → (x,t)
↳ Fix camera outside the system.
(Sample : 100%)

② Floating car Data (Vehicle Navigation System)
↳ (sample : small) ↳ LiDAR, RADAR

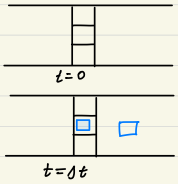
Eulerian

single loop detector → time of crossing & all.
double loop detector → vel. also

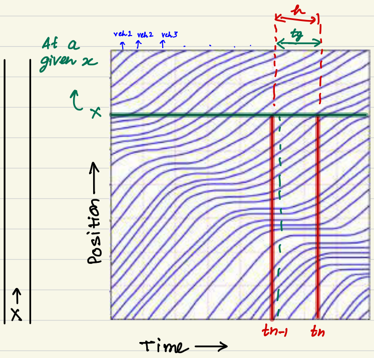


space is fixed

cross sectional data (loop detector)

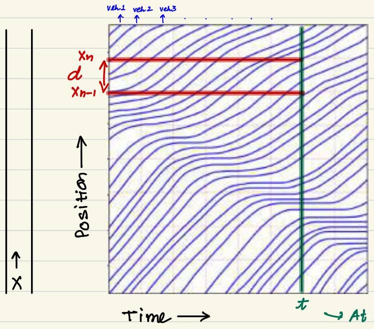


Finding traffic entities from vehicle trajectory data



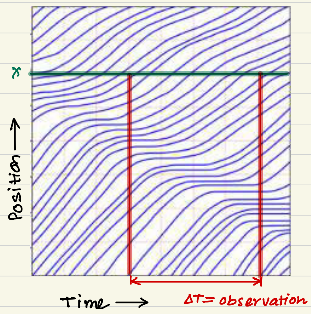
time headway $h = t_n - t_{n-1}$

time gap $t_g \approx t_n - t_{n-1}$



distance headway $= X_n - X_{n-1}$

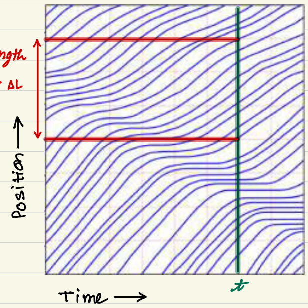
distance gap $\approx X_n - X_{n-1}$



No. of veh.

$$\text{Flow}, Q = \frac{\Delta N}{\Delta T} = \frac{6}{\Delta T}$$

ΔT = observation period.



Observed length ↪ ΔL

No. of veh.

$$\text{Density}, \rho = \frac{\Delta N}{\Delta L} = \frac{9}{\Delta L}$$

Single lane road...

$\Delta x \simeq 100\,m$

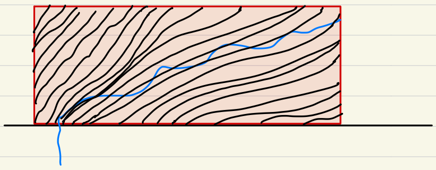
macroscopic models → For continuum hypothesis:

microscopically large (consider enough no. of vehicles)

macroscopically small (traffic dynamics does not vary significantly)

Double Loop Detector

Q →
ρ →
v →



Region : [0 s, 30 s] × [1000 m, 1200 m]

v_{TMS}

$$v_{SMS} = \frac{180\,m}{28\,s} = \quad \cdots \quad m/s$$

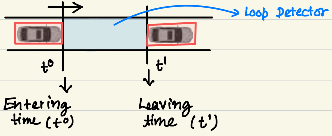
↳ In modelling we use the SMS speed mostly.

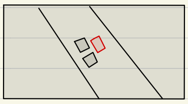
Loop Detectors

* Assume speed = constant

SINGLE LOOP DETECTOR

Measure the entry and exit times t^0 and t^1





$\text{Occupancy time} = t^1 - t^0$

$\text{time headway } (h) = t_n^0 - t_{n-1}^0$

entering time n^{th} vehicle

entering time $(n-1)^{th}$ vehicle

$\text{time gap} = t_n^0 - t_{n-1}^1$

speed → can't be calculated from single loop detector

DOUBLE LOOP DETECTOR

Composed of two (or more) induction loops seperated by a fixed distance.

distance b/w loops

$$\text{Vehicle speed}, v_i = \frac{\Delta x_{loops}}{(t_1^0)_{loop2} - (t_1^0)_{loop1}}$$

} → time difference b/w passing the two loops.

$\text{vehicle length} = v_i\,(t_i^1 - t_i^0)$

$\text{time headway} = h = t_i^0 - t_{i-1}^0$

Follower Leader

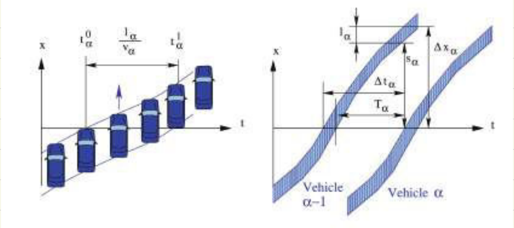
$\text{time gap b/w rear and front bumper} = t_i^0 - t_{i-1}^1$

$= h - \frac{V_{i-1}}{l_{i-1}}$

$\text{distance headway}, s = V_{i-1}\,h$ → time headway

$\text{distance gap b/w rear and front bumpers} = s - l_{i-1}$

distance headway



Single vehicle data as measured by an inductive loop detector.

TMS

Time Mean Speed (TMS)

$$v_{TMS} = \frac{\sum_{i=1}^{n} v_i}{n}$$

(x - fixed)

observation period = ΔT

- Arithmetic Mean Speed

SMS

Space Mean Speed (SMS)

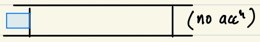
$$v_{SMS} = \frac{\text{Total Distance}}{\text{Total Time}}$$

$$v_{SMS} = \frac{\Delta x + \Delta x + \cdots + \Delta x}{\frac{\Delta x}{v_1} + \frac{\Delta x}{v_2} + \cdots + \frac{\Delta x}{v_n}} = \frac{n\,\Delta x}{\Delta x\,(\Sigma \frac{1}{v_i})}$$

$$v_{SMS} = \frac{n}{\sum_{i=1}^{n}\left(\frac{1}{v_i}\right)}$$

- Harmonic Mean Speed

☆ $V_{SMS} \leq V_{TMS}$, where equality holds if speeds are identical.



Δx

observed time = Δt

length = Δx

Δx — small

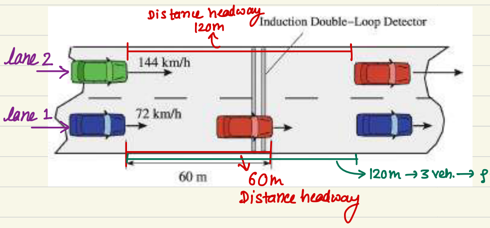
continuum hypo. holds

macros. large

micros. small

TRAFFIC DENSITY (ρ)

$\boxed{Q=\rho v}$ → Fundamental Relation in Traffic Flow Theory



→ 120m → 3 veh. → $\rho = \frac{3}{120} = \frac{1}{40}$ veh./m

↓

But $\rho_{calculated} = \frac{1}{45}\frac{veh}{m}$

This means calculated ρ is erroneous.

$$\rho(x,t) = \frac{Q(x,t)}{V(x,t)} = \frac{flow}{speed}$$

lane 1 = 144 kmph = 40 m/s ⟶ $d_1 = 120m$

lane 2 = 72 kmph = 20 m/s ⟶ $d_2 = 60m$

$h_1 = \frac{120m}{40m/s} = 3\,sec$ $\qquad h_2 = \frac{60m}{20m/s} = 3\,sec$

$Q_1 = \frac{1}{h_1} = \frac{1}{3}$ veh./sec. $\qquad Q_2 = \frac{1}{h_2} = \frac{1}{3}$ veh./sec.

Total Flow $= Q_{total} = Q_1 + Q_2 = \frac{2}{3}$ veh./sec.

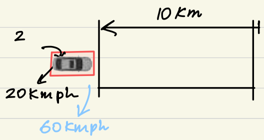
$$v_{TMS} = \frac{1}{20}(40+20) = 30\,m/s$$

$$\rho = \frac{Q_{total}}{v_{TMS}} = \frac{2}{3\times 30} = \frac{2}{90} = \frac{1}{45}\ veh./m$$

$$v_{SMS} = \frac{n}{\Sigma\left(\frac{1}{v_i}\right)} = \frac{2}{\frac{1}{20}+\frac{1}{40}} = 26.67\ m/sec$$

$$\rho = \frac{Q_{total}}{v_{SMS}} = \frac{2}{3\times 26.67} = \boxed{\frac{1}{40}}$$

→ It is matching with the actual value.



$TT_1 = 10$ min $\qquad$ 1 → 60 kmph

$TT_2 = 20$ min $\qquad$ 2 → 20 kmph

Total travel time = 40 mins

$$v_{TMS} = \frac{1}{2}(60+20) = 40\ kmph$$

$TT = 30$ min

$$v_{SMS} = \frac{2}{\frac{1}{60}+\frac{1}{20}} = 30$$

$$\frac{Total\ dist.}{v_{SMS}} = TT = \left(\frac{10+10}{30}\right) = \frac{20}{30}$$

Floating curve Data (FCD) ⟶ can find only speed, travel time, etc.

⟶ Intrinsic data can only be found

⟶ can't find density, flow, etc (extrinsic prop.)

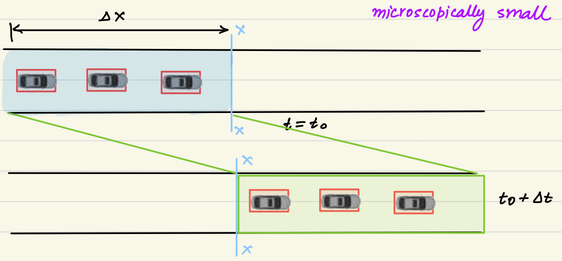
Trajectory Data ⟶ can find everything (Q, ρ, v, Travel time, etc)

* FCD → sample size is low.

* Traffic wave always travels — 15 kmph (universal speed)

Fundamental Relation

Δx → macroscopically large
microscopically small



$Q = \frac{\Delta N}{\Delta T}$ ———— ①

$\rho = \frac{\Delta N}{\Delta X} \Rightarrow \Delta N = \rho\,\Delta X$ ⟶ Put in ① above

$Q = \frac{\rho\,\Delta X}{\Delta T}$ ⟶ SMS

$\boxed{Q = \rho v}$ → This v is SMS (space mean speed)

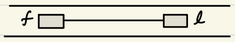
* True for higher models as well. Won't change with models.

$Q = \rho v$

$v = f(\rho)$

$Q = \underbrace{\rho f(\rho)}$

↪ converting into one variable.

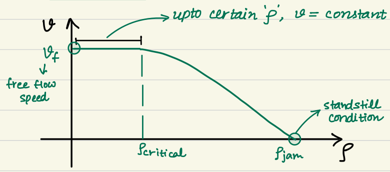


$v_f = f(d)$ $\qquad$ microscopic!

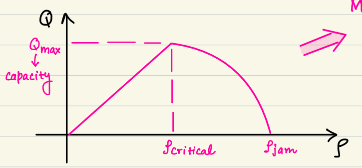
↓ $\qquad$ ↓ $\qquad$ converting to

$v = f(\rho)$ $\qquad$ macroscopic!

$\boxed{v \text{ vs } \rho}$ Speed Density Diagram



$\boxed{Q \text{ vs } \rho}$ Flow Density Diagram



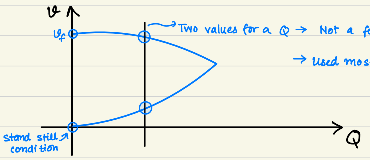
Mostly used for Traffic Flow Theory

$\rho_{cr} = 25$ veh./km

$v_{cr} = 90 - 80$ kmph

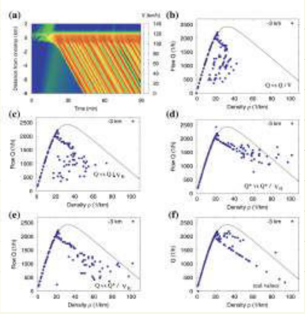
↪ critical speed → v for which we generally get the ρ_{cr} or max. speed.

$\boxed{v \text{ vs } Q}$ Speed Flow Diagram

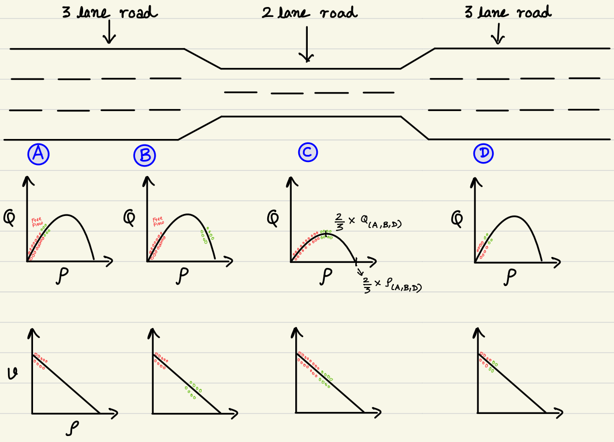


→ Two values for a Q → Not a function → Mathematically not a good plot.

→ Used mostly because both of them (v, Q) are easy to measure.



Lane changing



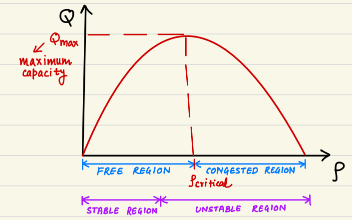
flow (2.5 lane)

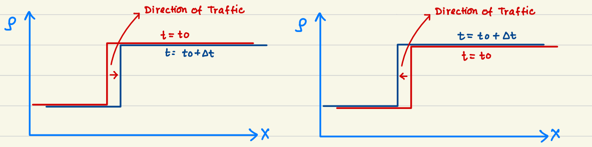
2.5 $Q_{capacity}$

flow $\int_{2\text{-lane}}^{0}$

2Q capacity

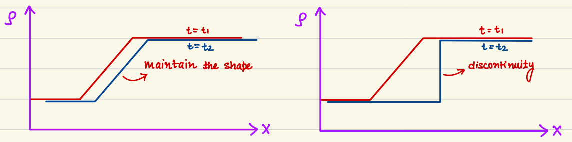
$$\boxed{\frac{Q_{3lane}}{3} = \frac{Q_{2lane}}{2}} \Rightarrow Q_{2lane} = \frac{2}{3} \times Q_{3lane}$$



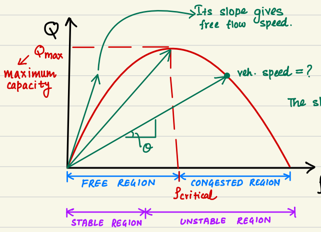


FREE REGION ($\rho < \rho_c$) CONGESTED REGION ($\rho > \rho_c$)

propogation of vel. = +ve propogation of vel. = −ve



STABLE REGION UNSTABLE REGION

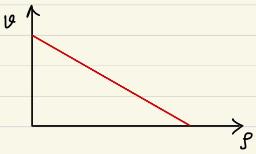


$Q = \rho v \Rightarrow v = \frac{Q}{\rho}$

The slope of this point from origin will give free flow speed.

$$v = \frac{\Delta Q}{\Delta \rho} = \frac{Q_2 - Q_1}{\rho_2 - \rho_1} = \frac{Q_2 - 0}{\rho_2 - 0} = \frac{Q_2}{\rho_2}$$

Green - Shield Model



$$\boxed{v = v_f\left(1 - \frac{\rho}{\rho_{max}}\right)}$$

$\rho = 0$; $v = v_f$

$\rho = \rho_{max}$; $v = 0$

$Q = \rho v$

$Q = \rho v_f\left(1 - \frac{\rho}{\rho_{max}}\right)$

$\rho = 0$; $Q = 0$

$\rho = \rho_{max}$; $Q = 0$

$$Q = v_f\left(\rho - \frac{\rho^2}{\rho_{max}}\right)$$

$$\frac{dQ}{d\rho} = v_f\left(1 - \frac{2\rho}{\rho_{max}}\right)$$

$$\frac{dQ}{d\rho} = 0 \Rightarrow 1 - \frac{2\rho}{\rho_{max}} = 0 \Rightarrow \frac{2\rho}{\rho_{max}} = 1 \Rightarrow \rho = \frac{\rho_{max}}{2}$$

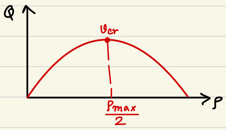
Characteristic speed, $c = \frac{dQ}{d\rho}$

$$v = \frac{v_f}{2}$$

$$v = v_f\left(1 - \frac{\rho}{\rho_{max}}\right) = v_f\left(1 - \frac{\rho_{max}}{2\rho_{max}}\right)$$

$$v = v_f\left(1 - \frac{1}{2}\right) = \frac{v_f}{2}$$

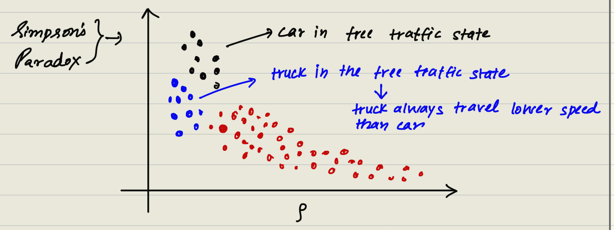
$$\therefore \boxed{v_{cr} = \frac{v_f}{2}}$$



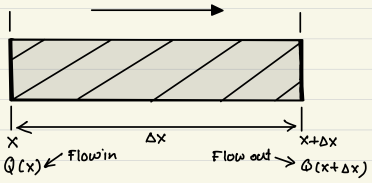
$$Q_{cr} = \rho_{cr} v_{cr} = \frac{\rho_{max}}{2} \times \frac{v_f}{2} = \frac{\rho_{max} v_f}{4}$$

Simpson's Paradox

It is a statistical phenomenon that tells that the data in different groups will show a trend but all the groups put together (data put together) that may show a different trend.



Continuity Equation (conservation of vehicles)



Number of vehicles, $N = \rho \Delta x$

Rate of change of vehicles inside the region $\Delta x = \text{Flow}_{in} - \text{Flow}_{out}$

$\text{Flow}_{in} = Q(x)$ $\text{Flow}_{out} = Q(x+\Delta x)$

Rate of change of vehicles inside the region $= \frac{\partial N}{\partial t} = \frac{\partial}{\partial t}(\rho \Delta x)$

$\Delta x \to$ very small, use Taylor's Series

$Q(x+\Delta x) = Q(x) + \frac{\partial Q}{\partial x}\Delta x \longrightarrow$ First order approximation $Q(x,t)$

$$\frac{\partial}{\partial t}(\rho \Delta x) = Q(x) - Q(x+\Delta x)$$

$$\frac{\partial}{\partial t}(\rho \Delta x) = \cancel{Q(x)} - \left(\cancel{Q(x)} + \frac{\partial Q}{\partial x}\Delta x\right)$$

$$\cancel{\Delta x}\frac{\partial \rho}{\partial t} = -\cancel{\Delta x}\frac{\partial Q}{\partial x}$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial Q}{\partial x}$$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} = 0}$$

conserved form!

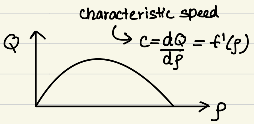
conservation of vehicles.

simple yet powerful concept.

$Q = f(\rho)$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial \rho}\frac{\partial \rho}{\partial x} = 0}$$

Non-conserved form!



$$\frac{d\rho}{dt} + \frac{\partial Q}{\partial \rho}\frac{d\rho}{dx} = 0$$

characteristic speed!

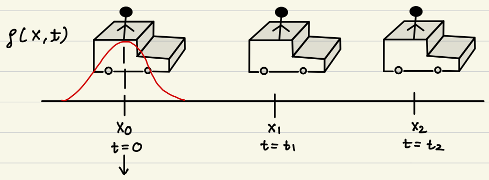
$$\frac{d\rho}{dt} + c(\rho)\frac{d\rho}{dx} = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0}$$

→ Non-linear first order ODE.

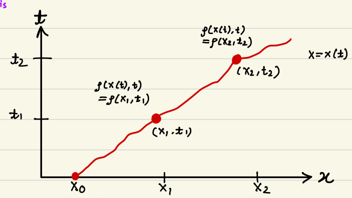
Characteristic

A curve in the x-t plane along which the solution can be easily defined.



$\rho(x,t) = \rho(x_0, 0)$

$= \rho(x_0)$ → usually written like this (convention)



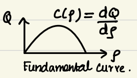
$\rho = \rho(x(t), t)$

$$\frac{d\rho}{dt} = \frac{\partial x}{\partial t}\frac{\partial \rho}{\partial x} + \frac{\partial \rho}{\partial t}\cdot\cancelto{1}{\frac{dt}{dt}}$$

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial \rho}{\partial x}$$

From Continuity Equation (conservation of vehicles)

$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0 \longrightarrow$ Compare with above eqⁿ



Fundamental curve.

$$\boxed{\frac{dx}{dt} = c(\rho) \quad \text{and} \quad \frac{d\rho}{dt} = 0}$$

ODE

Density is constant along the curve (t-x).

$\frac{dx}{dt} = c(\rho)$

$\int dx = \int c(\rho)dt$

$x = c(\rho)t + c_0$

At $t=0$, $x = x_0 \implies c_0 = x_0$

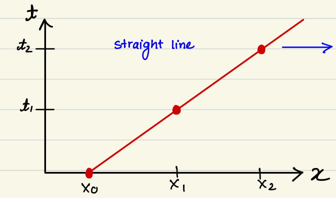
$\boxed{x = c(\rho)t + x_0}$

$x_0 = x - c(\rho)t$

$\frac{d\rho}{dt} = 0$

$\rho =$ constant

density can easily find from the characteristic curve



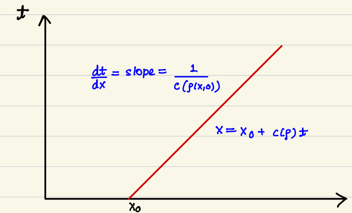
→ Along this curve, $\rho =$ constant.

$\rho = \rho(x,t) = \rho(x_0, 0) = \rho(x - c(\rho)t, 0)$

$\rho =$ constant

$\rho(x,t) = \rho(x_0)$

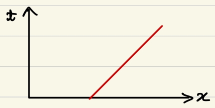
$\frac{dx}{dt} = c(\rho) =$ constant



✻ In Traffic Flow Theory, we'll use only straight line characteristics

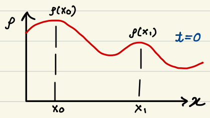
't' vs 'x' curve → Why 't' vs 'x' → For visualization, it is easy.

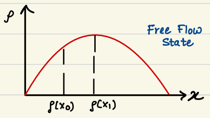
- Characteristic speed can have both +ve and -ve values.

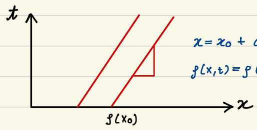


$c(\rho) > 0$

$c(\rho) < 0$

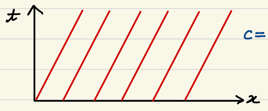






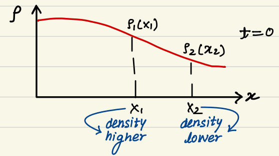
$x = x_0 + c(0)t$ → constant

$\rho(x,t) = \rho(x - c(\rho)t, 0) = \rho(x - c(\rho)t)$



c = constant

$\rho(x,t)$



$\frac{dQ}{d\rho_1} < \frac{dQ}{d\rho_2}$

$c(\rho_1) < c(\rho_2)$

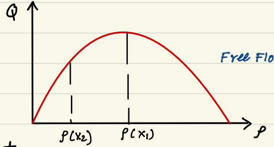
$\frac{1}{c(\rho_1)}, \frac{1}{c(\rho_2)}$

$x = x_0 + c(\rho)t$

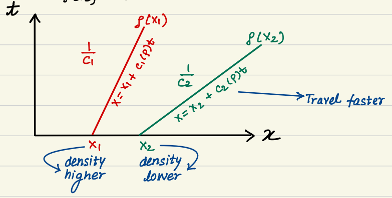
$c(\rho)t = x - x_0$

$t = \frac{x - x_0}{c(\rho)}$

$\frac{dt}{dx} = \frac{1}{c(\rho)}$



Free Flow



Fan like structure

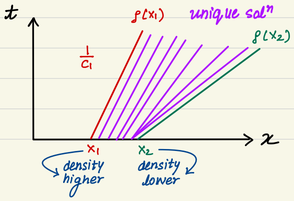
$\rho(x,t)$

$X = x(\gamma)$

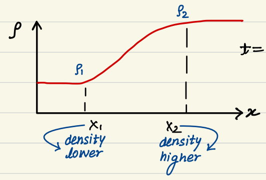
$t = t(0)$

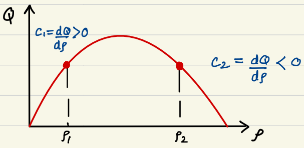
$t = \gamma$

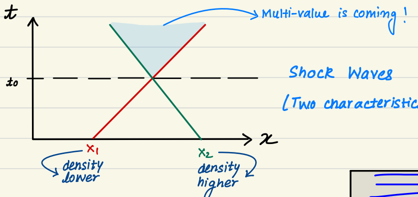
Expansion wave in Traffic Flow Theory.



- Expansion Flow in Traffic Flow Theory.
- It is making a fan like structure.

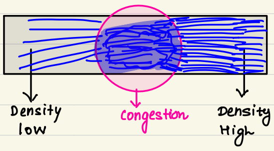


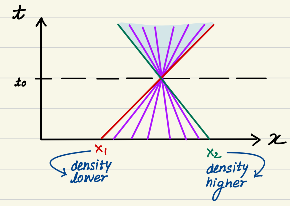




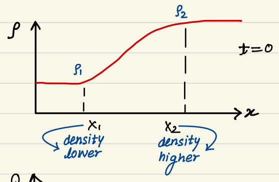
Shock Waves

(Two characteristics crossing each other)





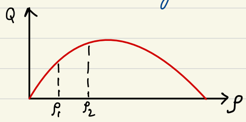
$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial \rho}\frac{\partial \rho}{\partial x} = 0}$$ → This formula won't be valid anymore here!

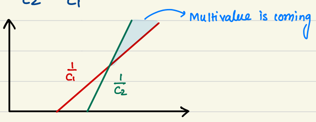


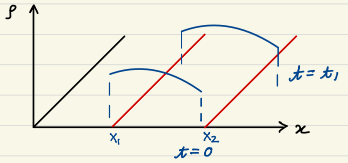
$\frac{dQ}{d\rho_2} < \frac{dQ}{d\rho_1}$

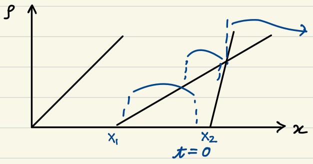
$C_2 < C_1$

$\frac{1}{C_2} > \frac{1}{C_1}$









$\frac{d\rho}{dx} \longrightarrow \infty$

↓

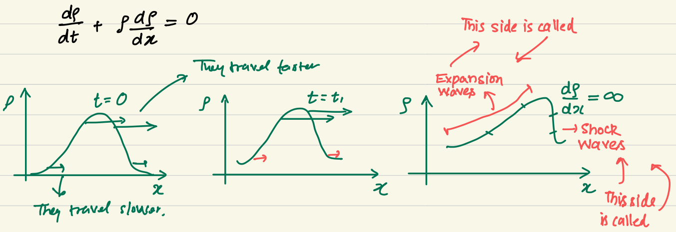
that's why the formula $\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial \rho}\frac{\partial \rho}{\partial x} = 0$

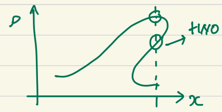
won't be valid anymore.

Example:-

$\frac{dQ}{d\rho} = \rho$ → characteristic speed.

$\frac{d\rho}{dt} + \rho\frac{d\rho}{dx} = 0$



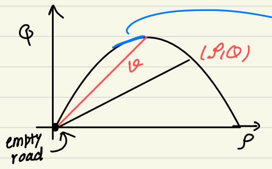


two values → not any more a function

$\rho(x_3)$ → Not a function → How to fix it?
↓
will teach in next class.

Repeat last Class

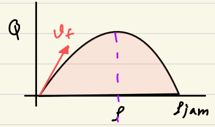
- Why 'c' always from origin. $v = \frac{Q}{\rho} = \frac{Q-0}{\rho-0}$ (Q: Flux, ρ: Density)



$c = \frac{dQ}{d\rho}$ → characteristic speed
↓
if you're creating small perturbations then how will it affect the traffic flow. → very small

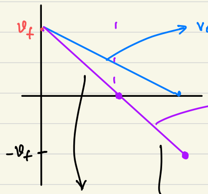
v = equilibrium speed (vehicle speed)

— all vehicle travelling same speed (want to travel)

- 

Green Shield's Model

$v = v_f\left(1 - \frac{\rho}{\rho_{max}}\right)$



→ vehicle speed ⤳ vehicle speed ≥ 0
↳ never goes below zero

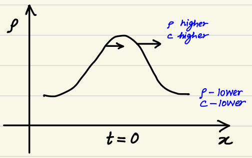
→ characteristics speed ⤳ it can take both +ve and −ve value.

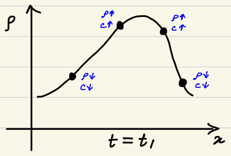
in dirⁿ of traffic
↓
in free state
c take +ve value

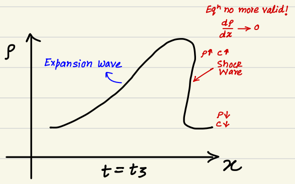
against dirⁿ of traffic
↓
in congested state
c take −ve value

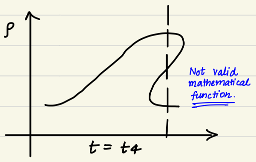
Gradient Catastrophe

small changes in the environment can cause the model to produce vastly different outputs, leading to unpredictable and potentially dangerous behavior.



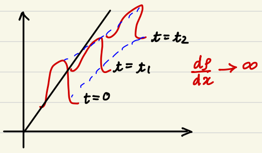




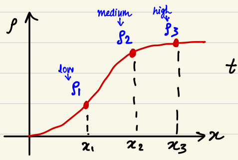


$$\boxed{\frac{\partial\rho}{\partial t} + \frac{\partial Q}{\partial\rho}\cdot\frac{\partial\rho}{\partial x} = 0}$$

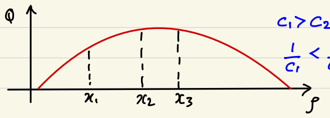
$\frac{dQ}{d\rho} = c =$ characteristic speed.



$\frac{d\rho}{dx} \rightarrow \infty$

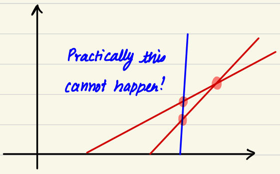


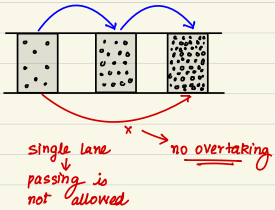
t=0



$c_1 > c_2 > c_3$

$\frac{1}{c_1} < \frac{1}{c_2} < \frac{1}{c_3}$





Example

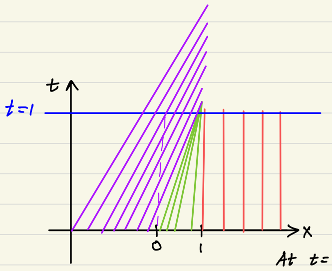
$$\frac{d\rho}{dt} + \rho\frac{d\rho}{\partial x} = 0 \qquad \frac{dQ}{d\rho} = \rho \rightarrow \text{characteristic speed.}$$

$$\rho(x,0) = \begin{cases} 1 & , \quad x < 0 \\ 1-x & , \quad 0 \leq x \leq 1 \\ 0 & , \quad x > 1 \end{cases} \quad x \in (-\alpha,\alpha) \quad t \in [0,\alpha)$$

$\rightarrow \frac{dQ}{d\rho} = \rho = 0 \searrow \frac{1}{0} = \infty$

$\frac{dQ}{d\rho} = \rho = 1 - x_0$

$= \frac{1}{1-x_0}$

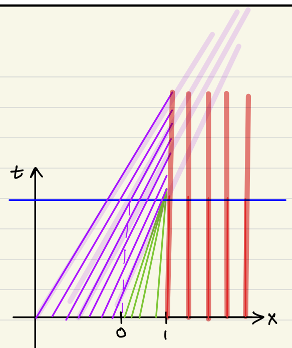


At t=1, all of the characteristics meet
↳ Breaking Time.

t=1 ⟶ Breaking Time

t<1 ⟶ smooth and unique solution.

t≥1 → multi-value



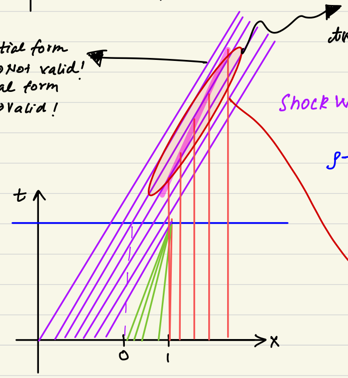
$$\boxed{\frac{d\rho}{dt} + \rho\frac{d\rho}{dx} = 0}$$

Differential

Conservation Form
↓

Doesn't hold when two characteristics meet.

Differential form
↳Not valid!
Integral form
↳Valid!

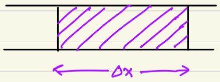


Shock Wave

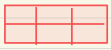
ρ → Piece-wise smooth function → bcoz
left side → cont.
right side → cont.

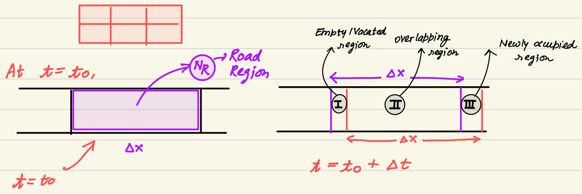
In this region,
ρ → discontinuous. function.

Lets' derive integral form of conservation of vehicles.



Vehicle System = N = Number of Vehicles





$N = \int \rho\, dx$

Rate of change of vehicles present in the system $= \frac{dN}{dt}$

$$\frac{dN}{dt} = \lim_{\Delta t \to 0} \frac{N|_{t_0+\Delta t} - N|_{t_0}}{\Delta t}$$

$$N|_{t+\Delta t} = N_{II} - N_{III}\Big|_{t_0+\Delta t}$$

$$N|_{t_0+\Delta t} = N_R - N_I + N_{III}\Big|_{t_0+\Delta t}$$

$$\frac{dN}{dt} = \lim_{\Delta t \to 0} \frac{(N_R - N_I + N_{III}|_{t_0+\Delta t} - N_R|_{t_0}}{\Delta t}$$

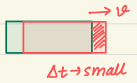
$$\frac{dN}{dt} = \underbrace{\lim_{\Delta t \to 0} \frac{N_R|_{t_0+\Delta t} - N_R|_{t_0}}{\Delta t}}_{\frac{\partial N_R}{\partial t}} + \lim_{\Delta t \to 0} \frac{N_{III}|_{t_0+\Delta t}}{\Delta t}$$

$$- \lim_{\Delta t \to 0} \frac{N_I|_{t_0+\Delta t}}{\Delta t}$$

$$\cancelto{0}{\frac{dN}{dt}} = \frac{\partial N_R}{\partial t} + \lim_{\Delta t\to 0}\frac{N_{\text{III}}|_{t_0+\Delta t}}{\Delta t} - \lim_{\Delta t\to 0}\frac{N_{\text{I}}|_{t_0+\Delta t}}{\Delta t}$$

$$0 = \frac{\partial N_R}{\partial t} + \lim_{\Delta t\to 0}\frac{N_{\text{III}}|_{t_0+\Delta t}}{\Delta t} - \lim_{\Delta t\to 0}\frac{N_{\text{I}}|_{t_0+\Delta t}}{\Delta t}$$

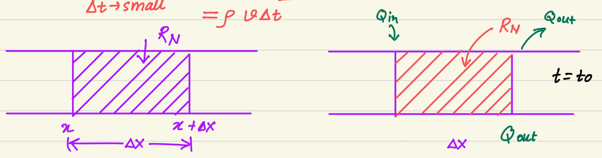
$$N_{\text{III}} = \underbrace{\int \rho\, dx}_{\text{Region III}}\Big|_{t_0+\Delta t}$$

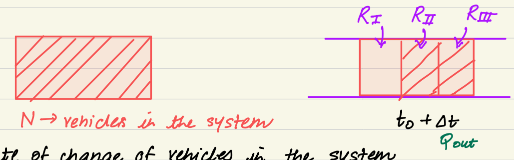


$= \rho\int dx$

$= \rho\ \Delta x_{\text{III}}$

$= \rho\ v\Delta t$





N → vehicles in the system

Rate of change of vehicles in the system

$$= \frac{dN}{dt} = \lim_{\Delta t\to 0}\frac{N|_{t_0+\Delta t} - N|_{t_0}}{\Delta t}$$

$$N|_{t_0+\Delta t} = N_{\text{II}} + N_{\text{III}}|_{t_0+\Delta t}$$

$$N|_{t_0+\Delta t} = N_R - N_{\text{I}} + N_{\text{III}}|_{t_0+\Delta t}$$

$$N_{\text{II}} = N_R - N_{\text{I}}$$

$$\frac{dN}{dt} = \lim_{\Delta t\to 0}\frac{\left[N_R - N_{\text{I}} + N_{\text{III}}|_{t_0+\Delta t} - N_R|_{t_0}\right]}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\frac{N_R|_{t_0+\Delta t}}{\Delta t} - \lim_{\Delta t\to 0}\frac{N_1|_{t_0+\Delta t}}{\Delta t}$$

$$N_{\text{III}} = \int \rho\, dx_{t_0+\Delta t}$$
$$= \text{region III}$$

$$\frac{dN}{dt} = \frac{dN_R}{dt} + \lim_{\Delta t\to 0}\frac{N_{\text{III}}|_{t_0+\Delta t}}{\Delta t} - \lim_{\Delta t\to 0}\frac{N_{\text{I}}}{\Delta t}$$

$$0 = \frac{\partial \int\rho\, dx}{\partial t} + \lim_{\Delta t\to 0}\frac{N_{\text{III}}|_{t_0+\Delta t}}{\Delta t}$$

$$N_R = \int\rho\, dx - \lim_{\Delta t\to 0}\frac{N_{\text{I}}|_{t_0+\Delta t}}{\Delta t}$$

$\lim_{\Delta t\to 0}\frac{N_{\text{III}}|_{t_0+\Delta t}}{\Delta t}$ ·

$N_{\text{III}}|_{t_0+\Delta t} = \int\rho\, dx\ (t_0+\Delta t)$

macroscopic speed

Δt = small v → same speed

ρ = constant

$N_{\text{III}}|_{t_0+\Delta t} = \rho\int dx$

$\int dx = \Delta x$

$N_{\text{III}}|_{t_0+\Delta t} = \rho\Delta x$

$\Delta x = v\Delta t$

$N_{\text{III}}|_{t_0+\Delta t} = \rho v\Delta t$

$$\lim_{\Delta t\to 0}\frac{N_{\text{III}}|_{t_0+\Delta t}}{\Delta t} = \frac{\rho v\Delta t}{\Delta t} = \rho v \hookrightarrow \text{Region III}$$
$$= Q_{out}$$

$$\lim_{\Delta t\to 0}\frac{N_{\text{I}}|_{t_0+\Delta t}}{\Delta t} = \frac{\rho v\Delta t}{\Delta t} = \rho v\ \text{region I} = Q_{\text{I}}$$

$$N_{\text{I}}|_{t_0+\Delta t} = \int\rho\, dx\ \ (\text{region I})$$

Δt → small

ρ → constant $= \rho\int dx$

v → constant $= \rho\Delta x = \rho v\Delta t$

$$0 = \frac{\partial}{\partial t}\int\rho\, dx + \underbrace{Q_{out} - Q_{in}}$$

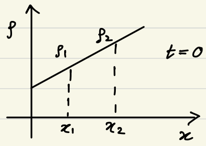
$$\boxed{\frac{\partial}{\partial t}\int\rho\, dx = Q_{in} - Q_{out}}$$

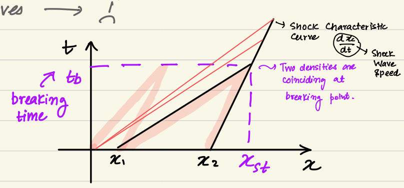
Final ↓ Integral Form !!

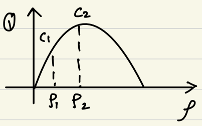
$$\frac{\partial\int\rho\, dx}{\partial t} = Q_{in} - Q_{out}$$

Shock wave

Coming back to Shock Waves → :(

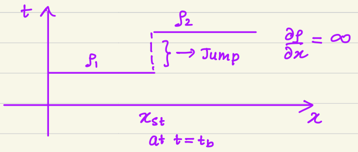






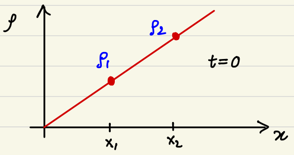
$C_1 > C_2$

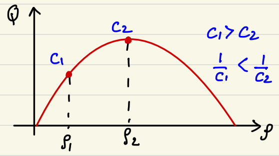
$\frac{1}{C_1} < \frac{1}{C_2}$



· Jump Discontinuity is there.

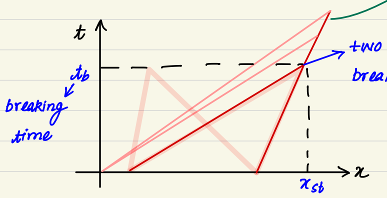
Shock Waves





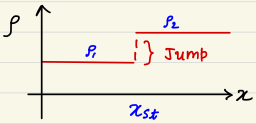
$C_1 > C_2$

$\frac{1}{C_1} < \frac{1}{C_2}$



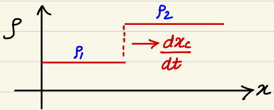
Shock characteristic curve $\frac{dx_c}{dt}$ → shock wave speed.

two densities are coinciding at the breaking point.



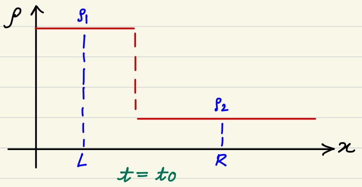
at $t = t_b$ (breaking time)

- $\frac{\partial\rho}{\partial x} = \infty$
- Jump Discontinuity is there



$t > t_b$ (breaking time)

Derivation of propagation velocity

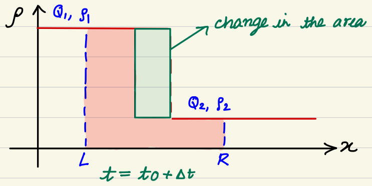


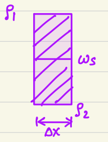
$$\frac{d\int_L^R \rho\,dx}{dt} = Q_{in} - Q_{out}$$

finite value or not?

$\int_L^R \rho\,dx$ = Area under the curve

Yes, this interval have finite quantity.





$\int \rho\,dx$ = Area = $(\rho_1 - \rho_2)\,\Delta x_s$

$\frac{d}{dt}\int \rho\,dx$ = change in area at time interval dt.

$\Delta x_s = w_s dt$

$$\frac{d}{dt}\int_L^R \rho\,dx = \text{rate of change of area}$$
$$= \frac{(\rho_1-\rho_2)\,\Delta x}{dt} = \frac{(\rho_1-\rho_2)\,w_s\,dt}{dt} = (\rho_1-\rho_2)\,w_s$$

$$\frac{d}{dt}\int \rho\,dx = (\rho_1-\rho_2)\,w_s \qquad \left\{ \because \frac{d}{dt}\int \rho\,dx = Q_{in} - Q_{out} \right\}$$

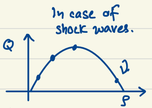
$$Q_{in} - Q_{out} = (\rho_1-\rho_2)\,w_s$$

$$\boxed{w_s = \frac{Q_{in} - Q_{out}}{\rho_1 - \rho_2}} \longrightarrow \text{shock wave speed} = \frac{dx_s}{dt}$$

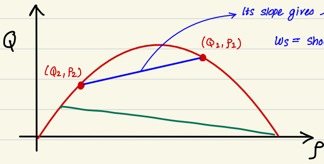
Propogation of shock waves.

- The propogation velocity of density variations $c(\rho) = \frac{dQ}{d\rho}$ is given by the slope of the fundamental diagram.
- The propogation velocity of shock fronts is given by the slope of the secant connecting points of the fundamental diagram corresponding to traffic on either side of the front.
- The vehicle speed $v = \frac{Q(\rho)}{\rho}$ is given by the slope of the secant connecting the origin with the corresponding point on the fundamental diagram.

In case of shock waves.

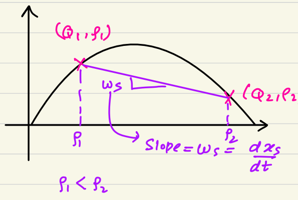


- $\frac{dQ}{d\rho}$ → characteristic speed → simulation
- When a shock wave occurs, perturbation moves forward.
- Vehicle Speed → Always travel forward
  Characteristic speed → Both ways (forward + backward)
- Shock Wave ⇒ Different Traffic States not just 1.
  1 traffic state ⇒ No shock wave
  >1 traffic states ⇒ shock waves
- Shock wave – imaginary / virtual boundary b/w two traffic state
  – not a real boundary
  – virtual – moves with traffic.



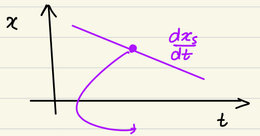
Its slope gives shock wave speed.

w_s = shock wave speed = slope between two traffic states.

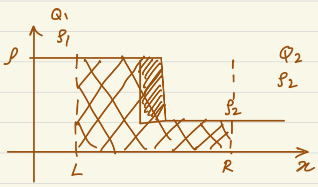


Slope $= w_s = \frac{dx_s}{dt}$

$\rho_1 < \rho_2$



No change in slope. as compared in the fundamental diagram



$\frac{d}{dt}\int \rho\,dx$

$\int \rho\,dx$ = Area

$\frac{d}{dt}\int \rho\,dx$ = change in area at time interval dt.

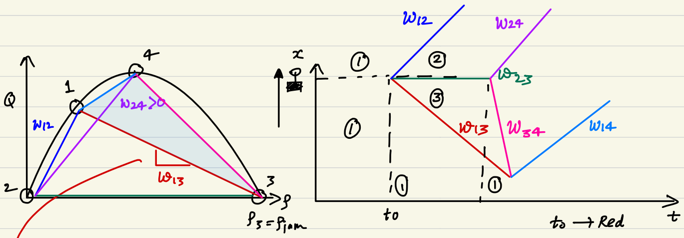
Area $= (\rho_1 - \rho_2)\,\Delta x_s$ $\qquad$ $\Delta x_s = w_s dt$

$$\frac{d}{dt}\left(\int \rho\,dx\right) = \frac{(\rho_1 - \rho_2)\,w_s\,dt}{dt} \qquad (\rho_1-\rho_2)\,w_s = Q_1 - Q_2$$

$$\boxed{\frac{d}{dt}\int \rho\,dx = Q_{in} - Q_{out}} \qquad \boxed{w_s = \frac{Q_1 - Q_2}{\rho_1 - \rho_2}}$$

Shock wave speed $= \frac{dx_s}{dt}$

- Shockwaves
- Construct congestion
- Dynamics
- Identify different traffic states.



$$\frac{dx_{13}}{dt} = w_{13} = \frac{Q_1 - Q_2}{\rho_1 - \rho_2}$$

$$w_{12} = \frac{Q_1 - Q_2}{\rho_1 - \rho_2} > 0$$

LWR
↓
First order model

$$\boxed{Q = \rho v}$$

$$w_{13} = \frac{Q_1 - Q_2}{\rho_1 - \rho_3} < 0$$

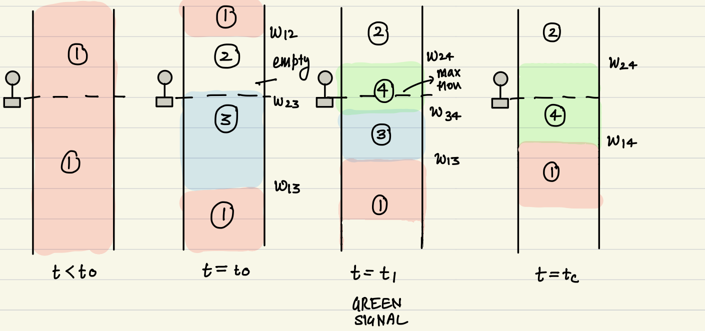
$$w_{23} = \frac{Q_2 - Q_3}{\rho_2 - \rho_3} = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial t}\frac{d\rho}{dx} = 0$$

↓
congestion dynamics

$$w_s = \frac{Q_{in} - Q_{out}}{\rho_1 - \rho_2}$$

vehicle speed
↓
I measure it from origin.
↓
Different from shock wave speed



t < to $\qquad$ t = to $\qquad$ t = t_1 $\qquad$ t = t_c

GREEN SIGNAL

vehicle speed – I measure it from origin to that point in fund. diagram.
– Different from shock wave speed.

Traffic Phenomenon → empirical data

Shock wave Theory
First Order Model

$$\begin{cases} Q = \rho v \\ \frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} = 0 \end{cases}$$

Q, ρ, v

$w_{id} = \frac{Q_1 - Q_d}{\rho_i - \rho_d}$

$\boxed{v = f(\rho)}$ ← functional form

$Q = \rho v$

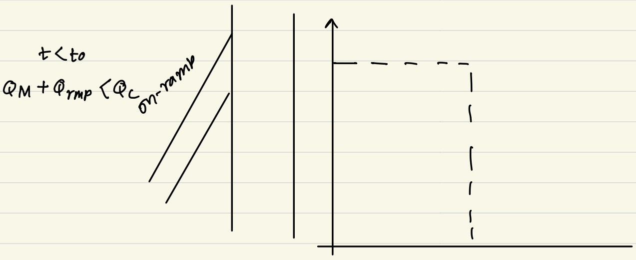
$Q = \rho f(\rho)$

Greenshield Model

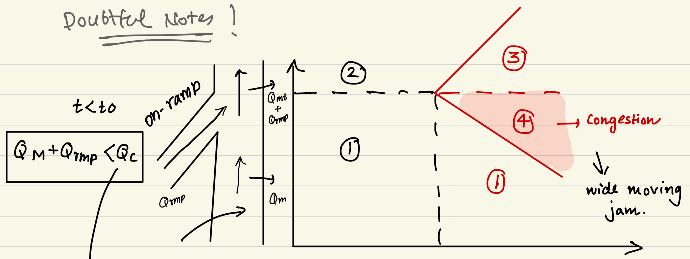
$v = v_f\left(1 - \frac{\rho}{\rho_{max}}\right)$

$\frac{\partial \rho}{\partial t} + \frac{\partial g(\rho)}{\partial x} = 0$

macroscopic data
Q, ρ, v,
↓
loop detectors



Doubtful Notes !

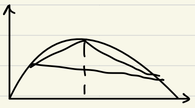


Capacity of single lane road.

t>to

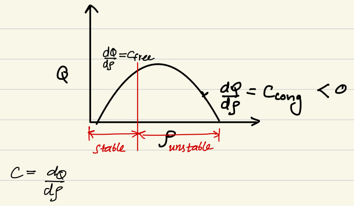
$Q_m + Q_{rmp} > Q_c$

flow less romb.



occurence of moving jams

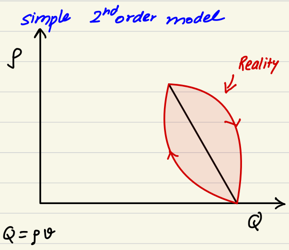
Growing amplitude of perturbations $= A(t) = A_0 \cdot e^{\sigma(t-t_0)}$



- What does sign of growth rate imply?
- Critical flow
- Free flow
- Jam upstream of bottleneck

Traffic Hysteresis

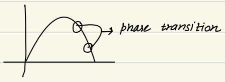
simple 2nd order model



$Q = \rho v$
$Q = f(\rho)$ ⟶ equilibrium model
$Q = \rho f(\rho)$
$Q = g(\rho)$

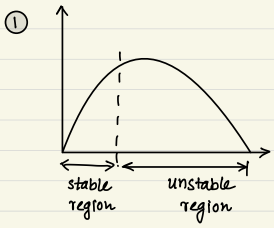
Triangle Fundamental Diagram

Equilibrium model.



① Anticipation
② Reaction Time } Higher order model.

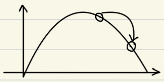
①



⟶ wide moving jam

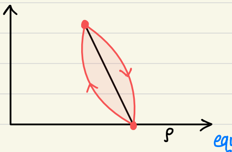
② Phase Transitions

LWR does not capture phase transitions.
↓
First order model



③ LWR — does not capture growing pertubations.
④ LWR — congested (or) jam → always shows uniform congestion dynamics.

Traffic hysteresis



First order model

$Q = v_e$

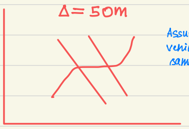
$Q = \rho v_e$

$v_e = v_f\left(1 - \frac{\rho}{\rho_{max}}\right)$ ↳ parabolic FD (Fund. Diagram)

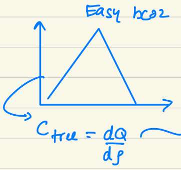
equilibrium speed ↓ Assume each vehicle travel same speed.

$Q = \rho f(\rho)$

$Q = g(\rho)$



Easy bcoz its linear and $C_{cong} = C_{free}$ (same slope)



$C_{cong} = \frac{dQ}{d\rho}$ → same speed bcoz of same slope.

$C_{free} = \frac{dQ}{d\rho}$

Anticipation behaviour

$$f(x+\Delta) = f(x) + \Delta \frac{df}{dx} + O(\Delta^2) + \cdots$$

$$f(x+\Delta_x, y+\Delta_y) = f(x,y) + \Delta_x \frac{\partial f}{\partial x} + \Delta_y \frac{\partial f}{\partial y} + \cdots$$

macroscopic speed $v(x,t) = v(\rho(x,t))$

$\rho(x,t)$
↳ $\rho(x+\Delta,t)$

anticipation

$$v(x,t) = v(\rho(x+\Delta,t))$$

$$\boxed{\rho(x+\Delta,t) = \rho(x,t) + \Delta \frac{\partial f}{\partial x}}$$

$f(x+\Delta,y)$

$$v(x,t) = v\Big(\underset{\uparrow}{\rho(x,t)} + \Delta \frac{d\rho}{\partial x}\Big)$$

$f(x+\Delta)$

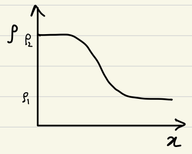
$$\boxed{v(x,t) = v(\rho(x,t)) + \Delta \frac{\partial \rho}{\partial x} \frac{dv}{d\rho}}$$

→ Bcoz speed = $f(\rho)$

v (↓) decrease — $(\Delta > 0)$ $(+ve)$ $(-ve)$

v (↑) increase — $\parallel$ $(-ve)$ $(-ve)$

$Q = \rho v$ $\quad$ $Q = \rho v_e$



$v < v_e$

$Q = \rho v$

$\tau \to$ Average Reaction Time

$$v(x,t) = v(\rho(x,t))$$

↓ This is driver maintaining speed.

$x \to f(t)$ → including reacⁿ time

$x(t+\tau)$

$$= v(\underline{x(t+\tau)}, t+\tau) \qquad x(t+\tau) = x(t) + \tau \frac{dx}{dt} = x(t) + \tau v$$

$$= v(x(t) + \tau v, t+\tau)$$

$$= v(x,t) + \tau v \frac{\partial v}{\partial x} + \tau \frac{\partial v}{\partial t}$$

$$v + \tau v \frac{\partial v}{\partial x} + \tau \frac{\partial v}{\partial t} = v(\rho)$$

↑ equilibrium speed

$v(\rho) = v_e(\rho)$

$$v = v_f\left(1 - \frac{\rho}{\rho_{max}}\right)$$

$$v = f(\rho)$$

$$\tau v \frac{\partial v}{\partial x} + \tau \frac{\partial v}{\partial t} = v_e(\rho) - v$$

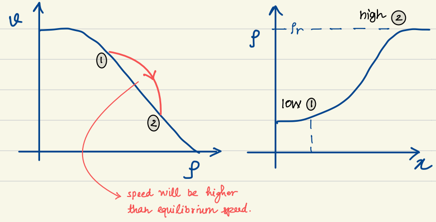
$$\tau\left(v \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t}\right) = v_e(\rho) - v$$

→ Final Formula

$$\boxed{\underbrace{\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x}}_{\text{Acceleration Term}} = \underbrace{\frac{v_e(\rho) - v}{\tau}}_{\text{Relaxation Term}}}$$

→ left term is acceleration bcoz of chain rule

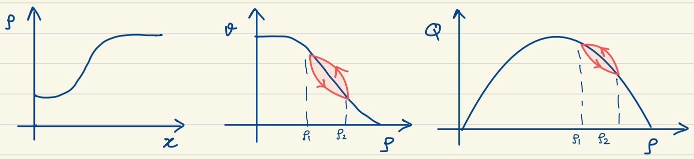
$$a = \frac{d\,v(x,t)}{dt} = v\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} \quad \text{(chain rule)}$$



anticipation

$$v(x,t) = v(\rho(x+A,t)) \quad (+)(+) \quad (-)$$

$$= v(\rho(x,t)) + \Delta \frac{\partial \rho}{\partial x} \frac{dv}{d\rho}$$



Reaction Time

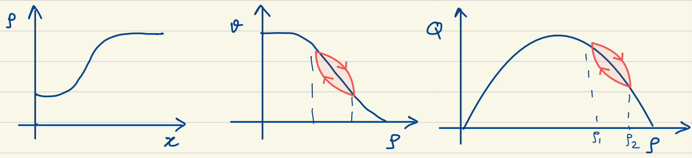
$$v(x(t+\tau), t+\tau) = v_e(\rho)$$

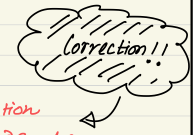
$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \frac{v_e(\rho) - v}{\tau}$$

$v \cancel{=} v_e(\rho)$

$v < v_e(\rho)$

↓ low speed → acceleration





Reaction time

$$v(x(t+\tau), t+\tau) = v(\rho(x+\Delta, t))$$

← anticipation

acceleration equation

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = v(\rho(\cancel{x},t)) + \frac{\Delta}{\tau}\frac{\partial \rho}{\partial x}\frac{dv}{d\rho}$$

↑ accⁿ

$$+ \frac{v_e(\rho) - v}{\tau}$$

← Payne model

First order model

$Q = \rho v$

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

$v = f(\rho)$

} LWR (00)

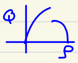
$Q = \rho v$

$$\frac{\partial \rho}{\partial t} + \frac{\partial v}{\partial x} = 0$$

① stable and unstable region

② phase transition

③ capacity drop ↪



Microscopic Model

$i \to$ follower

$i-1 \to$ Leader

Macroscopic — collection of vehicles

microscopic — driver-vehicle unit

ODE: $\underbrace{\frac{dx_i}{dt}}_{} = v_i$ $\qquad$ $\underbrace{\frac{dv_i}{dt}}_{\text{acc}^n} = f(x_i, x_{i-1}, v_i, v_{i-1})$

speed ← ; i-vehicle index

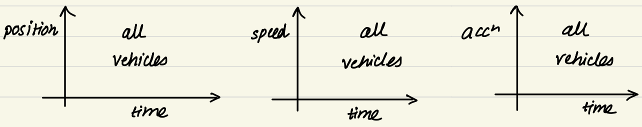
position of follower (x_i); position of leader (x_{i-1}); speed of follower (v_i); speed of leader (v_{i-1})

Two type of model :- → Longitudinal dynamics → We are doing this only!

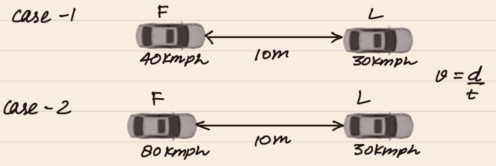
① Car Following — (accelerate, de-accelerate)

~~② lane changing~~ — (one lane to another lane) → lateral dynamics

output



*In model, we take time gap (and not space gap) bcoz time gap gives same gap for different speed.



$v = \frac{d}{t}$

Relative speed = speed difference between leader & follower
$= v_{i-1} - v_i$

Complete car following model

① represent free traffic
② " steady state
③ " dynamic situations

→ represent real acceleration profile
→ desired speed (v_0)

Steady State →

All vehicle travel at same speed and maintain same gap.

Acceleration = 0
→ ① maintain some safety gap.

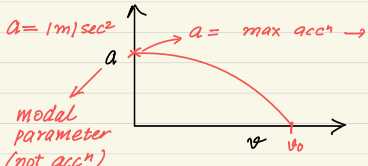
desired time gap → ② follow the leader with plausible time gap.

car following model
(acc)
↳ modal output

① Free Regime
② Steady State (equilibrium state)
③ Dynamic Situations

FREE REGIME EMPTY ROAD (stop condition)

$a = 1 m/sec^2$



a = max accⁿ → given by user

modal parameter (not accⁿ)
↓
user give this value

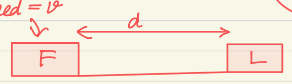
desired speed (v_0)
acc = 0
1) realistic acc profile
2) desired speed

STEADY STATE

① Minimum gap (s_0) ← Standstill condition

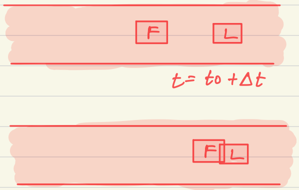
time gap (or) space gap

speed = v



time gap

$T = \frac{d}{v}$
↳ desired time gap.
≈1.5 sec



t= to +Δt

t= to + 2Δt

distance ↳ $S = S_0 + \underbrace{vT}$ → distance gap
standstill condition

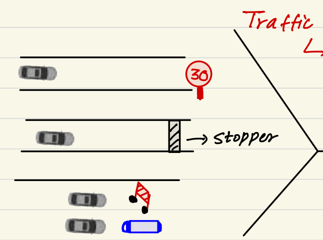
$T = \frac{d}{v}$ → distance, → speed

desired time gap.

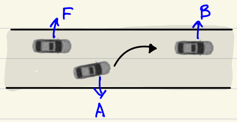
---

close-in
→ comfortable breaking behaviour
→ comfortable deceleration
1.5 m/sec²

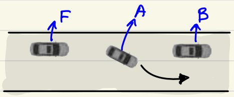
Traffic Obstacles
↳ Breaking comfortably



→ All the situations come under obstacles



cutting in
Earlier, B – leader
then A – leader



cutting out
Earlier A – leader
then B – leader

IDM (Intelligent Driver Model)

$$\dot{v} = a\left(1 - \left(\frac{v}{v_0}\right)^{\delta} - \left(\frac{s'}{s}\right)^{2}\right)$$

social force

model output

v = current speed
v_0 = desired speed
a = maximum acceleration

s' = desired gap
s = actual gap
δ = model parameter

$$\underbrace{a\left(1 - \left(\frac{v}{v_0}\right)^{\delta}\right)}_{\text{attractive part}} \qquad \underbrace{-a\left(\frac{s'}{s}\right)^{2}}_{\text{repulsive part}}$$

attractive part ↓ accelerate

repulsive part ↓ decelerate

$$s' = s_0 + vT + \frac{v\,\Delta v}{2\sqrt{ab}}$$

desired time gap

minimum gap (standstill condition)

$\Delta v = v - v_l$

follower speed

leader speed

b = comfortable deceleration

empty road $v = 0$ $s \to \infty$

$$\dot{v} = a\left(1 - \left(\frac{v}{v_0}\right)^{\delta} - \cancel{\left(\frac{s'}{s}\right)^{2}}\right)$$

0

s = actual gap ↓ $s \to \infty$

$$\dot{v} = a\left(1 - \left(\frac{v}{v_0}\right)^{\delta}\right)$$ For Empty Road

$$\dot{v} = a\left(1 - \cancel{\left(\frac{0}{v_0}\right)}\right)$$

0

$\dot{v} = a$

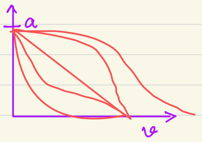
$\therefore \boxed{\dot{v} = a}$

Simple, Accident Free Model ⇒ That's why all researchers and scientists of autonomous vehicles use this model for test.

$v_0 = 60 \qquad v = 30$

$\dot{v} = a\left(1 - \frac{30}{60}\right)$

$\dot{v} = a\,(1 - 0.5) = 0.5a$



$$\dot{v} = a\left(1 - \left(\frac{v}{v_0}\right)^8 - \left(\frac{s'}{s}\right)^2\right)$$

$$a\left(1 - \left(\frac{v}{v_0}\right)^8\right) \; - \; a\left(\frac{s'}{s}\right)^2$$

Attraction — Repulsion

s' = desired distance

s = actual distance

v = current speed

v_0 = desired speed

δ_1 → a model parameter = max. acc

Case-1 : Empty Road starting from standstill condition

$s \to \infty$

$\frac{s'}{s} \to 0$

$$\dot{v} = a\left(1 - \left(\frac{v}{v_0}\right)^8\right)$$

Case-1 $\underline{\delta = 1}$

$v = 0$

$\dot{v} = a(1 - 0)$

$\dot{v} = a$

Case-2 $v = 30$ kmph $\qquad v_0 = 60$ Kmph

$\dot{v} = a\left(1 - \left(\frac{1}{2}\right)\right)$

$\dot{v} = a(1 - 0.5) = 0.5$

Case-3 : $v = 60$ kmph

$v_0 = 60$ kmph

$\dot{v} = a\left(1 - \left(\frac{60}{60}\right)\right)$

$\dot{v} = 0$

Steady state : Homogenous and Equilibrium state

$v = v_e$

$S \to$ constant

$s' = s_0 + vT + \cancel{\frac{v\Delta v}{2\sqrt{ab}}}$ 0

$s' = s_0 + vT$

near steady state

$v < v_e$

$v \to v_e$

$\Delta v = v - v_e$

↳ negligible

Assume $\underline{\underline{\Delta v}} \to 0$

$$\dot{v} = a\left(1 - \cancel{\left(\frac{v}{v_0}\right)^\delta} - \left(\frac{s'}{s}\right)^2\right)$$

0 Assuming it zero (0).

Case-1 : $v < v_e$ ⟶ Steady State

desired gap $= s' = s_0 + vT = 5m$

actual gap $= s = 10m$

$v = a\left(1 - \left(\frac{s'}{s}\right)^2\right)$

$$s' = \underbrace{s_0 + vT}_{\text{equilibrium term.}}$$

$s' = 5m$
↓
driver want to accelerate
$v < v_e$

$v = a\left(1 - \left(\frac{5}{10}\right)^2\right)$

$= a\,(1 - (0.5)^2)$

$= 0.75\,a$

$v > v_e$

desired gap $= 10m$

actual gap $= 5m$

$v = \left(1 - \left(\frac{s'}{s}\right)^2\right) = a\left(1 - \left(\frac{10}{5}\right)^2\right)$

$= a\,(1 - 4)$

$= -3a$

① Emerging

② Normal

$$S = \underbrace{S_0 + \cancel{v}^{\,0}}_{\text{equilibrium term}} + \underbrace{\frac{v\Delta v}{2\sqrt{ab}}}_{\text{dynamic term}}$$

equilibrium term ≪ dynamic term

Incomplete

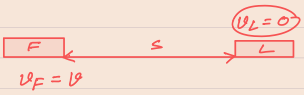
$$v = -a\left(\frac{v\Delta v}{2s\sqrt{ab}}\right)^2$$

$\Delta v = v - v_l$
$\Delta v = v$

$$= -a\left(\frac{v\Delta v}{2s\sqrt{ab}}\right)^2 = -a\,\frac{v^4}{4abs^2}$$

$$= -\frac{v^4}{4s^2b} = -\frac{\left(\frac{v^2}{2s}\right)^2}{b}$$

Breaking Distance



$v^2 = u^2 + 2as$
$v \rightarrow$ FS $\rightarrow 0$
$u \rightarrow$ IS $\rightarrow v$
$s \rightarrow$ Actual ~~gap~~

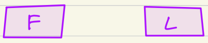
$-v^2 = 2bs$

$$\boxed{b_k = \frac{v^2}{2s}}$$

$$\dot v = -\frac{b_{km}^2}{b}$$

Comfortable deceleration

$$\boxed{\dot v = -\frac{b_{km}^2}{b}}$$



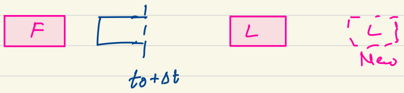
Case-1 : Very critical

$t = t_0$

$b_{km} > b$
$(4) > (2)$

$$v = \frac{-4\times 4}{2} = -8$$

-4 would have worked but follower applies -8. Hurry.



$t_0 + \Delta t$

$b_{km} \rightarrow$ lesser

$b_{km}\left(\frac{4}{3}\right)$

$t = t_0 + 2\Delta t$

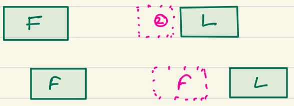
comfortable

① Normal

$b > b_{km}$
(4) (2)

$$v = -\frac{4}{4}$$

$$v = -1$$



$t = t_0 + \Delta t$
(3)

$t_0 + 2\Delta t$

comfortable decleration

Linear Stability Analysis

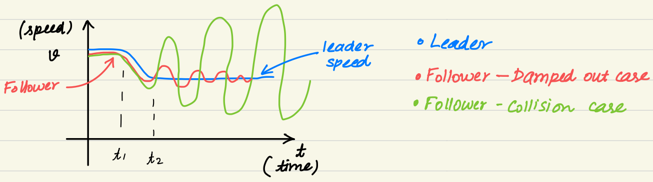
- Whenever you're proposing a new model, you're to do it.
- If you create small pertubations.
- Homogenous and Steady State

All vehicle travel same speed.

$v, a = 0$, $s =$ constant
$v_e =$ equilibrium speed.
$s_e =$ equilibrium gap.

$s = s_e + y$ perturbation
$v = v_e + u$

$y \ll s_e$
$u \ll v$



Decomposing into steady state and perturbation.

$s(t) = s_e + y(t)$
$v(t) = v_e + u(t)$

local linear stability analysis.
↓
Considering only leader & follower.

$v_l = v_e$

General car Following Model

$$\frac{dx}{dt} = v_i \longrightarrow \text{speed eq}^n \longrightarrow \underline{\underline{X}}$$

$$\frac{dv_i}{dt} = f(s, v, v_l) \longrightarrow \text{acc}^n\ \text{eq}^n$$

$$s_i = x_{i-1} - x_i - L_{i-1}$$

$$\frac{ds_i}{dt} = \frac{dx_{i-1}}{dt} - \frac{dx_i}{dt} - \cancel{\frac{dL_{i-1}}{dt}}^{\,0}$$

$$\boxed{\frac{ds_i}{dt} = v_{i-1} - v_i} \longrightarrow \checkmark$$

$$\boxed{\frac{dv_i}{dt} = f(s_i, v_i, v_l)}$$

$$s_i = s_e + y(t)$$

$$\frac{ds_i}{dt} = \cancel{\frac{d(s_e)}{dt}}^{\,0} + \frac{dy(t)}{dt}$$

$$\boxed{\frac{ds_i}{dt} = \frac{dy}{dt}}$$

$$\frac{dy}{dt} = v_{i-1} - v_i \qquad v_{i-1} = v_l = v_e$$

$$\frac{dy}{dt} = v_e - (v_e + u) \qquad v_i = v_e + u$$

$$\boxed{\frac{dy}{dt} = -u}$$ ——①

$y << S_e$
$u << v$

$$\frac{dv_i}{dt} = f(s, v, v_e)$$

$$v_i = v_e + u$$

$$\frac{dv_i}{dt} = \cancelto{0}{\frac{dv_e}{dt}} + \frac{du}{dt}$$

$$\frac{dv_i}{dt} = \frac{du}{dt}$$

$$= f(S_e + y, v_e + u, v_e)$$

Taylor series!!

$$= \cancelto{0}{f(S_e, v_e, v)} + \left.\frac{\partial f}{\partial s}\right)_e y + \left.\frac{\partial f}{\partial v}\right)_e u + \left.\frac{\partial f}{\partial v_e}\right)_e \cancelto{0}{(v_e - v_e)}$$

acc^n in steady state is zero

$$= \left.\frac{\partial f}{\partial s}\right)_e (s - s_e) + \left.\frac{\partial f}{\partial v}\right)_e (v - v_e)$$

$$\boxed{\frac{du}{dt} = \frac{\partial f}{\partial s} y + \frac{\partial f}{\partial v} u} \text{ ——— ②}$$

$$\frac{dy}{dt} = -u \text{ ——— ①}$$

$$\frac{du}{dt} = \frac{\partial f}{\partial s} y + \frac{\partial f}{\partial v} u \text{ ——— ②}$$

$$\frac{du}{dt} = f_s y + f_v u \text{ ——— ③}$$

Diff. one more time

$$\frac{d^2y}{dt^2} = -\frac{du}{dt} = -(f_s y + f_v u)$$

$$\frac{d^2y}{dt^2} = -f_s y - f_v u \quad \rightarrow -\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} = -f_s y - f_v\left(-\frac{dy}{dt}\right)$$

$$\frac{d^2y}{dt^2} = -f_s y + f_v \frac{dy}{dt}$$

$$\boxed{\frac{d^2y}{dt^2} - f_v \frac{dy}{dt} + f_s y = 0}$$

$$\boxed{\frac{d^2y}{dt^2} - f_v \frac{dy}{dt} + f_s y = 0}$$

$$y = ce^{\lambda t}$$ → Eiler inverted form

$$\frac{dy}{dt} = c\lambda e^{\lambda t}$$ → substitute

$$\frac{d^2y}{dt^2} = c\lambda^2 e^{\lambda t}$$

$$c\lambda^2 e^{\lambda t} - f_v(c\lambda e^{\lambda t}) + f_s(ce^{\lambda t}) = 0$$

$$ce^{\lambda t}(\lambda^2 - \lambda f_v + f_s) = 0$$

$$\boxed{\lambda^2 - \lambda f_v + f_s = 0}$$ → Characteristic Equation.

$$\lambda_{1,2} = \frac{f_v \pm \sqrt{f_v^2 - 4f_s}}{2}$$

general solution of this equation (Linear ODE)

$$\boxed{y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}$$

Case 1: Real Roots

$\lambda_1, \lambda_2 < 0$ → Model Locally Stable

(Bcoz $\frac{dy}{dt}$ ↓ with time)

$f_v < 0$ $\quad f_s > 0$

$$\frac{f_v^2}{4} - f_s > 0$$

Case 2: Real Repeated Roots

$$\lambda_1, \lambda_2 = \frac{f_v}{2}$$

$$\boxed{y(t) = c_1 e^{\lambda_1 t} + t c_2 e^{\lambda_2 t}}$$

$f_v < 0$

→ model locally stable.

Case 3: Imaginary Roots

$$\lambda_1, \lambda_2 = \frac{f_v}{2} \pm i\omega$$

$$y(t) = e^{f_v/2} \underbrace{(c_1 \cos\omega t + c_2 \sin\omega t)}_{\text{sinusoidal function}}$$

$f_v < 0$ → Model locally stable

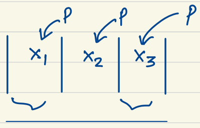
Stochastic Process

Random Process
over period of time
observing random

Bernoulli Process

Bernoulli Trial

Success	$x_i = 1$	head	vehicle arrival
Failure	$x_i = 0$	tail	vehicle not arrival



① Independent (One x_3 doesn't affect x_1)
② Time Homogenity (Probability is same)

Joint Prob mass function $P(x_1, x_2, x_3) = P(x_1) \cdot P(x_2) \cdot P(x_3)$
Joint mass time

upto time 't' → single Event
n − trial
K − success

$$\boxed{P(n, K) = (^nC_K)\, p^k (1-p)^k}$$ → Binomial Distribution. ↳ Discrete time

no. of time slot

K=1
K=2

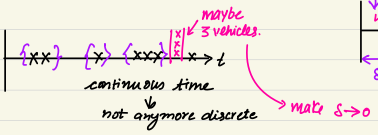
probability of success of each interval

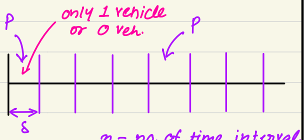
n time interval $= \frac{t}{8}$

$\Delta t = 10$ sec

Continuous time

→ Poisson Process





n = no. of time interval

Discrete Time

- Continuous time → Poisson Process
- Discrete time → Bernoulli Process

Assumptions :-
① Independent
② Time homogenity
③ $\delta \to 0$
④ $\lambda = \frac{\text{Probability}}{\text{Time Interval}}$

$P(K=1, \delta) = \lambda\delta$
$P(K=0, \delta) = 1 - \lambda\delta$
$P(K>1, \delta) = 0$

Bernoullis Process ≈ Poisson process

n = no. of interval.

p = probability of success.

np = no. of vehicles arriving (duration of time t)

$\frac{t}{\delta}(\lambda\delta)$ = no. of vehicles arriving (t)

$$np = \frac{t}{\delta}(\lambda\delta) = t\lambda \implies \lambda = \frac{np}{t}$$

$$\boxed{\lambda = \frac{np}{t}} \text{ // } \quad \text{or} \quad \boxed{p = \frac{\lambda t}{n}} \text{ //}$$

$$P(K,t) \approx P(n,K)$$

K,t → continuous time n,K → discrete time

$$P(K,t) \approx P(n,K)$$

$\delta \to 0$

$n \to \infty$

continuous process

$$= ({}^nC_K)\, p^K (1-p)^{n-K}$$

$$= \frac{n!}{(n-K)!\,K!}\left(\frac{\lambda t}{n}\right)^K\left(1-\frac{\lambda t}{n}\right)^{n-K}$$

$$= \frac{n\times(n-1)\times(n-2)\times\cdots\cancel{(n-K)\times\cdots\times 1}}{\cancel{((n-K)\times\cdots\times 1)}\times K!}\left(\frac{\lambda t}{n}\right)^K\left(1-\frac{\lambda t}{n}\right)^{n-K}$$

$$= \frac{n\times(n-1)\times(n-K+1)}{K!}\left(\frac{\lambda t}{n}\right)^K\left(1-\frac{\lambda t}{n}\right)^n\left(1-\frac{\lambda t}{n}\right)^{-K}$$

$$= \cancel{\frac{n}{n}}\cdot\cancel{\frac{n-1}{n}}\cdots\cdots\cancel{\left(\frac{n-K+1}{n}\right)}\frac{(\lambda t)^K}{K!}\left(1-\frac{\lambda t}{n}\right)^n\left(1-\frac{\lambda t}{n}\right)^{-K}$$

$$= 1 \lim_{n\to\infty}\frac{(\lambda t)^K}{K!}\left(1-\frac{\lambda t}{n}\right)^n\left(1-\frac{\lambda t}{n}\right)^{-K}$$

$$= \frac{(\lambda t)^K}{K!}\lim_{n\to\infty}\left(1-\frac{\lambda t}{n}\right)^n\lim_{n\to\infty}\left(1-\frac{\lambda t}{n}\right)^{-K}$$

$$= \frac{(\lambda t)^K}{K!}e^{-\lambda t}$$

$$\boxed{P(K,t) = \frac{e^{-\lambda t}(\lambda t)^K}{K!}}$$

---

arrival

M/M/1 Queue

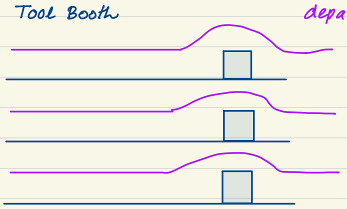
M/M/1 → How many no. of server

M/M/1 → departure / service

M – Memoryless property → poisson process

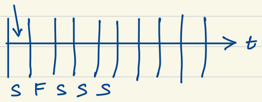
Tool Booth

departure (or) service → poisson process



arrival rate = 'λ' veh./sec

departure rate = H veh/sec.

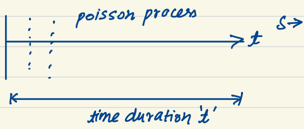


N = number of intervals (or) number of trail

n = number of success = number of vehicle arrivals

$$P(s) = \frac{n}{N}$$

probability vehicle arrival at each interval



$\delta \to 0$

$N = \frac{t}{\delta}$ $n = \lambda t$

$$P = \frac{\lambda t}{\frac{t}{\delta}} \implies \boxed{P = \lambda\delta}$$

Probability of vehicle arrival (δ)

$\boxed{\Delta t = \delta}$ → * In book used same.

Probability of departure = $H\delta$

$$\begin{cases} P(A=1) = \lambda\delta \\ P(A=0) = 1-\lambda\Delta t \end{cases}$$

one interval

$P(D=1) = H\Delta t$

$P(D=0) = 1-H\Delta t$

$P(A=2) = 0$

expected arrival ← λ = mean arrival rate

expected departure ← H = mean departure (or) service rate

Assumption $\boxed{\frac{\lambda}{H} < 1 \longrightarrow \text{Effective Queue}}$

λ = 5 veh/hour

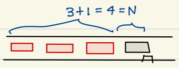
$\boxed{12\text{mins}}$ → one vehicle arrival → average value / expected value

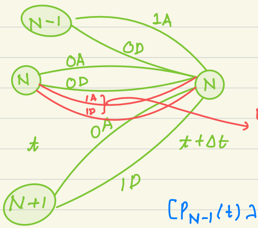
H = 6 veh/hr

10 mins → one vehicle departure

$\frac{\lambda}{H} > 1$ → not effective Queue

N = Number of vehicle in the system at time $t+\Delta t$ {Queue + service}





System

Not possible X

More than 2 events can't occur bcoz of poisson process.

$[P_{N-1}(t)\lambda\Delta t]$ → omitting Δt^2 term ← $P_{N-1}(t)H\Delta t$

$$P_N(t+\Delta t) = P_{N-1}(t)\times\lambda\Delta t\times(1-H\Delta t) + P_{N+1}(t)(1-\lambda\Delta t)H\Delta t + P_N(t)(1-\lambda\Delta t)(1-H\Delta t)$$

→ $P_N(t) - P_N(t)\lambda\Delta t - P_N H\Delta t$

$\Delta t \to 0$

$\cancel{\Delta t}$ very small

$$P_N(t+\Delta t) = P_{N-1}(t)\lambda\Delta t + P_{N+1}(t)H\Delta t + P_N(t) - P_N\lambda\Delta t - P_N H\Delta t$$

$$P_N(t+\Delta t) - P_N(t) = P_{N-1}(t)\lambda\Delta t + P_{N+1}(t)H\Delta t - P_N\lambda t - P_N H\Delta t$$

$$\lim_{\Delta t\to 0}\frac{P_N(t+\Delta t) - P_N(t)}{\Delta t} = P_{N-1}(t)\lambda + P_{N+1}(t)H - P_N\lambda - P_N H$$

$$\frac{dP}{dt} = \lambda P_{N-1}(t) + H P_{N+1}(t) - \lambda P_N - H P_N$$

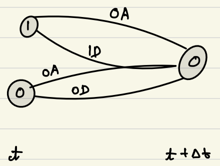
Steady state (Assuming prob. doesn't change)

(dropping time component)

$$\frac{dP}{dt} = 0 = \lambda P_{N-1} + H P_{N+1} - \lambda P_N - H P_N$$

$$\boxed{\lambda P_{N-1} + HP_{N+1} = \lambda P_N + HP_N} \text{ ——— ①}$$

Going to extreme case ⟶ 0 vehicles in the system



$$P_0(t+\Delta t) = P_1(t)(1-\lambda\Delta t)H\Delta t + P_0(t)(1-\lambda\Delta t)(1-\cancel{H\Delta t})$$

$$P_0(t+\Delta t) = P_1(t)H\Delta t + P_0(t) - P_0\lambda\Delta t$$

$$P_0(t+\Delta t) - P_0(t) = P_1(t)H\Delta t - P_0\lambda\Delta t$$

$$\lim_{\Delta t\to 0}\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = HP_1(t) - \lambda P_0$$

$$\frac{dP_0}{dt} = HP_1(t) - \lambda P_0$$

steady state ⟶ omit time term again

$$\frac{dP_0}{dt} = 0 = HP_1(t) - \lambda P_0 \Rightarrow HP_1(t) = \lambda P_0$$

$$\boxed{P_1(t) = \frac{\lambda}{H}P_0} \text{ ——— ②}$$

N = 1 in eqⁿ ① $\boxed{\lambda P_{N-1} + HP_{N+1} = \lambda P_N + HP_N}$

$$\lambda P_0 + HP_2 = \lambda P_1 + HP_1 \quad\rightarrow HP_1 = \lambda P_0$$

$$\lambda P_0 + HP_2 = \lambda P_1 + \lambda P_0$$

$$HP_2 = \lambda P_1$$

$$\boxed{P_2 = \frac{\lambda}{H}P_1} \rightarrow \text{Put } P_1 = \frac{\lambda}{H}P_0$$

$$P_2 = \frac{\lambda}{H}\times\frac{\lambda}{H}P_0 = \left(\frac{\lambda}{H}\right)^2 P_0$$

$$P_3 = \left(\frac{\lambda}{H}\right)^3 P_0$$

$$P_4 = \left(\frac{\lambda}{H}\right)^4 P_0$$

$$P_N = \left(\frac{\lambda}{H}\right)^N P_0$$

$P_1 = \rho P_0$

$P_2 = \rho^2 P_0$

$\vdots$

$P_N = \rho^N P_0$

P_0 = Probability of zero vehicle in the system.

Total No. of vehicle

$$0, 1, 2, \ldots\ldots \infty$$
$$\downarrow P_0 \;\; \downarrow P_1 \;\; \downarrow P_2 \qquad \downarrow P_\infty$$

$$\sum_{i=0}^{\infty} P_i = 1$$

$$= P_0 + \rho P_0 + \rho^2 P_0 + \rho^3 P_0 + \cdots$$

$$= P_0(1 + \rho + \rho^2 + \rho^3 + \cdots)$$

Assumption $\frac{\lambda}{H} < 1$ → GP

$$= P_0\left(\frac{1}{1-\rho}\right) = \frac{P_0}{1-\rho}$$

$$\frac{P_0}{1-\rho} = 1 \Rightarrow \boxed{P_0 = 1-\rho}$$

$P_0 = 1-\rho$

$P_1 = \rho(1-\rho)$

$P_2 = \rho^2(1-\rho)$

$P_3 = \rho^3(1-\rho)$

$\vdots$

$P_N = \rho^N(1-\rho)$

Derived the model

now, evaluate the system!

What to do?

Need of some variables :-

① waiting length

② total no. of veh. in the system

① Waiting Time

② Total no. of vehicle in the system.

1) Expected total number of vehicle in the system

X → Rv

$$E(X) = \sum x_i P_i$$

no. of veh — corresponding prob.

X=0 → P = 0.25

X=1 → P = 0.75

X=2 → P = 0.2

$$L_s = \cancel{0P_0}^{\,0} + 1P_1 + 2P_2 + 3P_3 + \cdots$$

↪ Expected number of veh. in the system.

$$L_s = P^1 + 2P_2 + 3P_3 + \cdots$$
$$= \rho P_0 + 2\rho^2 P_0 + 3\rho^3 P_0 + \cdots$$
$$= \rho P_0(1 + 2\rho + 3\rho^2 + \cdots)$$
$$= \rho P_0 \sum_{i=1}^{n} i\rho^{i-1}$$
$$= \rho P_0 \sum \frac{d}{d\rho}(\rho^i) \Rightarrow \rho P_0 \frac{d}{d\rho}(\Sigma \rho^i)$$

Expected Number of vehicles in the system

Number of vehicles

$$L_s = E[X]\; 0$$
$$= 0P^0 + 1P^1 + 2P^2 + 3P^3 + \cdots$$
$$= P^1 + 2P^2 + 3P^3 + \cdots$$
$$= \rho P_0 + 2\rho^2 P_0 + 3\rho^3 P_0 + \cdots$$
$$= \rho P_0[1 + 2\rho + 3\rho^2 + \cdots]$$
$$= \rho P_0 \Sigma i\rho^{i-1}$$
$$= \rho P_0 \Sigma \frac{d}{d\rho}(\rho^i)$$
$$= \rho P_0 \frac{d}{d\rho}(\Sigma \rho^i)$$

$X = 1 + 2\rho + 3\rho^2 + \cdots$ ①

$\rho X = \rho + 2\rho^2 + 3\rho^3 + \cdots$ ②

$X = \frac{1}{(1-\rho)^2}$ $\rho < 1$

$$= \rho P_0 \frac{d}{d\rho}\left(\frac{1}{1-\rho}\right)$$

$$= \rho P_0 \cdot \frac{1}{(1-\rho)^2} \quad\leftarrow \text{Put } \boxed{P_0 = 1-\rho}$$

$$= \frac{\rho(1-\rho)}{(1-\rho)^2}$$

$$= \frac{\rho}{(1-\rho)} = \frac{\lambda/H}{(1-\lambda/H)} = \frac{\lambda}{H-\lambda}$$

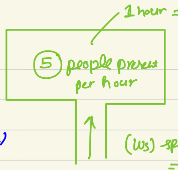
$$\boxed{L_s = \frac{\lambda}{H-\lambda}}$$

Expected waiting time

$$\boxed{W_s = \frac{L_s}{\lambda}}$$

(expected no. of veh. in system)

(mean arrival time)



1 hour ⇒ 5 people.

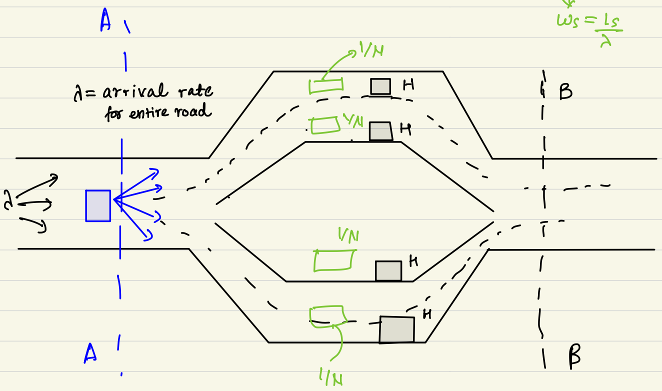
⑤ people present per hour

$\lambda = 10 \, \frac{people}{hr}$

(W_s) spending time
$= 30 \, min$
$= 0.5 \, hour.$

$$\boxed{L_s = \lambda W_s}$$

$$W_s = \frac{L_s}{\lambda}$$



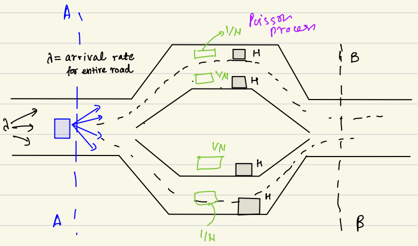
M/M/1 Queue
M/M/N Queue
λ = mean arrival rate of toll plazza
H = mean service rate (or) departure rate (toll both)

λ = number of lanes
$P_t = \frac{1}{N}$ (equally likely)

Assumption
Choosing lane also independent.

arrival $\lambda \rightarrow$ poisson process

$\lambda \rightarrow$ independent (poisson process)
$\lambda \rightarrow$ independent (poisson process)
$\lambda \rightarrow$ indep. " "
$\lambda \rightarrow$ indep. " "



$$W_t = \frac{1}{H-\lambda}$$
$$= \frac{1}{H-\frac{\lambda}{N}}$$

$$\boxed{W_\ell = \frac{N}{NH-\lambda}}$$

$$\boxed{L_s = \frac{N\lambda}{NH-\lambda}}$$

Arrival rate of particular lane $= \frac{\lambda}{N}$

$$\rightarrow L_\ell = \frac{\lambda}{H-\lambda}$$
$$= \frac{\lambda/N}{H-\lambda/N}$$
$$= \frac{\lambda}{NH-\lambda} \rightarrow \text{particular lane}$$

$$W_s = \frac{N}{NH-\lambda}$$

$$W_s \leq \Omega$$

Traffic engineers will design
$\Omega(\uparrow)$ — traffic not effective
$\Omega(\downarrow)$ — traffic effective

$$\frac{N}{NH-\lambda} \leq \Omega$$
$$N \leq \Omega(NH-\lambda)$$
$$N \leq \Omega NH - \Omega\lambda$$
$$\Omega\lambda \leq \Omega NH - N$$
$$N \geq \frac{\Omega\lambda}{(\Omega H-1)}$$

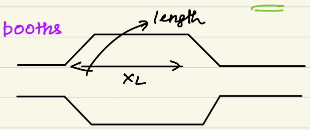
$N \geq \frac{\Omega\lambda}{\Omega H-1} \longrightarrow$ Take minimum

$$N = \left[\frac{\Omega\lambda}{\Omega H-1}\right]$$

Choose upper side !!

N = number of toll booths

$X_L(\downarrow)$ = spill over



L = vehicles in the system (only one lane)

$$P = P^0 + P^1 + P^2 + \cdots + P^L$$
$$= (1-\rho) + \rho(1-\rho) + \rho^2(1-\rho) + \cdots + \rho^L(1-\rho)$$
$$= 1-\cancel{\rho}+\cancel{\rho}-\cancel{\rho^2}+\cancel{\rho^2}-\cancel{\rho^3}\cancel{+}\cdots+\cancel{\rho^L}-\rho^{L+1}$$
$$= 1-\rho^{L+1}$$

$\boxed{P_L = 1-\rho^{L+1}} \leftarrow$ single lane probability. P_T

$P_T = (1-\rho^{L+1})^N \longrightarrow$ Each lane L vehicle.

P(at least one lane exceeding L vehicles)
$$= 1 - P_T$$
$$= 1 - (1-\rho^{L+1})^N$$